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Wave Propagation on Log-Periodic Transmission Lines

WILLIAM J. WELCH

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WAVE PROPAGATION ON LOG-PERIODIC
TRANSMISSION LINES

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WAVE PROPAGATION ON LOG-PERIODIC TRANSMISSION LINES

William J. Welch

1. ABSTRACT.

The propagation of waves along a transmission line, loaded log-periodically with impedance elements, is considered. By definition, the element spacing forms a geometric series, and the element magnitudes are proportional to their respective distances from the input end of the line. It is shown that the over-all effective impedance per unit length and admittance per unit length are represented by functions of the form: $l(x) = l(x\tau)$, where τ is the geometric ratio for the line. A general orthogonal series expansion is then found for such functions. Next, the general solution of the transmission line equations is considered for impedance and admittance per unit length of the form $l(x)$. It is shown that for a closely related set of equations, namely for impedance and admittance per unit length, of the form $l(x)/x$, the solution is of the form $x^\mu l(x)$. Although the solution for the purely log-periodic line cannot be factored into this simple form, a variational method is presented by means of which it can be approximately solved in terms of the logarithmic waves, x^μ . The variational technique makes use of formulas being stationary with respect to the current and voltage distributions on the line. The distributions are assumed to be of the form x^μ , and the stationary property allows one to calculate μ .

2. INTRODUCTION.

In recent years, the so-called 'log-periodic' type of antenna has received considerable attention, chiefly because of its great bandwidth capabilities. Many structures having the characteristic log-periodic geometry but of otherwise widely differing shapes have been experimentally investigated and found practical. However, although many common features have emerged from the various experiments, there exists at present no really adequate theoretical description capable of uniting these various features. In the present study we are attempting to gain understanding by considering a somewhat simpler problem, namely, the propagation of waves on a log-periodically loaded transmission line. This report contains the preliminary results of this study.

2. -- Continued.

In order to be clear about what constitutes a log-periodically loaded transmission line, consider, as an example, the line shown in Figure 1.

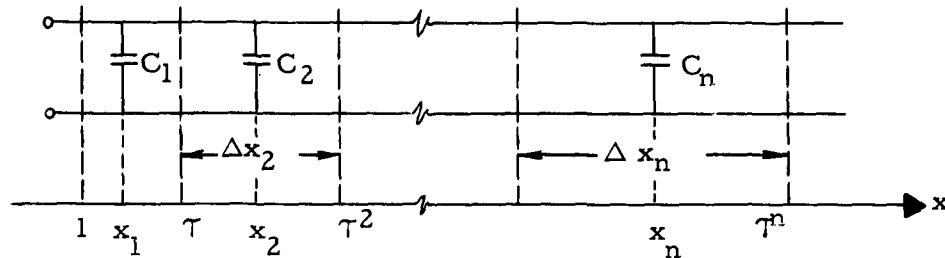


Figure 1

A Log-periodically Loaded Transmission Line

This consists of a uniform line with lumped capacitances connected across the line, their locations forming a geometric progression. The ratio of the positions of any two neighboring capacitors is a fixed constant τ , i.e., $x_n/x_{n-1} = \tau$. In addition, the magnitude of each capacitor is proportional to its distance from the origin ($x = 0$), and thus the ratio of any two neighboring capacitors is the same fixed constant, $\tau : C_n/C_{n-1} = \tau$. Let us now divide the line into cells or segments, each containing one capacitor and having the length Δx_n . The requirement that the ratio of the lengths of any neighboring segments be the same constant τ permits a simple description of the entire line in terms of any one of its segments or cells. That is, each successive cell may be obtained from its smaller neighbor by an expansion of everything in the cell by the fixed factor τ . This statement also defines the log-periodic transmission line in the more general case in which the cells may contain lumped inductances and resistances as well as capacitances. Of course, the general shape of a log-periodic antenna will fit this description. It is divided into cells, with each cell obtainable from its smaller neighbor by expansion by a factor τ .

Let us now consider how we may best represent the impedance and admittance per unit length of a log-periodic transmission line as a function of x , the coordinate along the line. In the example of Figure 1, the capacitance per unit length, including the lumped capacitors C_n , may be represented by the curve of Figure 2.

2. -- Continued.

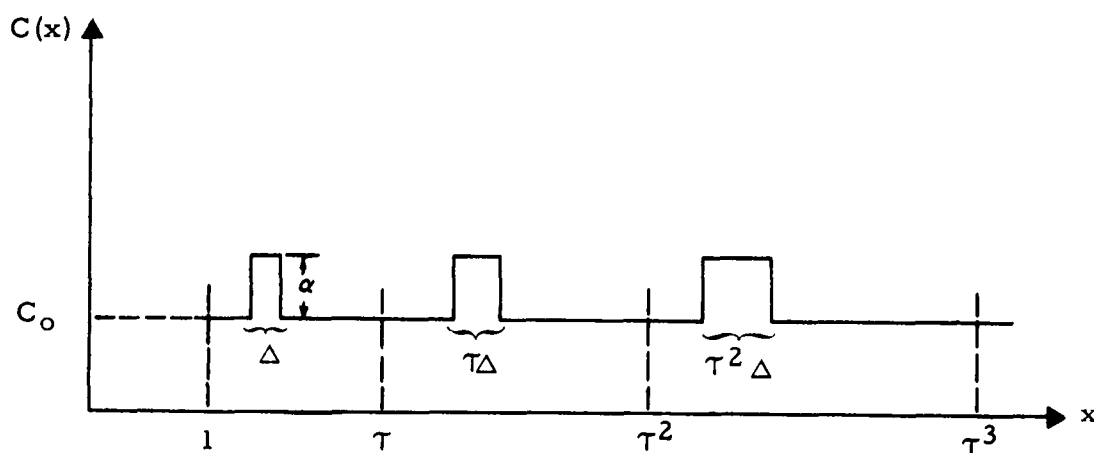


Figure 2

Capacitance per Unit Length on a
Log-periodic Transmission Line

C_0 is the per-unit-length capacitance of the unloaded line. $C_1 = \alpha \Delta$ is the lumped element in the first cell. The lumped capacitance in the second cell is evidently τ times the one in the first cell, and so on. Although the curve of Figure 2 provides the proper total 'lumped' capacitance in each cell, it clearly represents these 'lumped' elements as distributions. However, the distribution nicely satisfies the condition of cell elements stretching from cell to cell and, in fact, better represents a log-periodic line than the truly lumped-element version of Figure 1.

In Figure 2, $C(x)$ has the property that $C(x) = C(x\tau)$. We shall designate all functions that satisfy this condition as log-periodic functions and generally denote them by the symbol $l(x)$; i. e., $l_1(x)$, $l_2(x)$, etc.

$$l(x) = l(x\tau) \quad (1)$$

2. -- Continued.

The most general log-periodic transmission line will be loaded by resistive and inductive as well as by capacitive elements, and will be described by impedances and admittances per unit length which are log-periodic functions. The problem thus becomes one of solving for the voltage and current distributions on a non-uniform line,

$$\frac{dV}{dx} = Z(x) I \quad ; \quad \frac{dI}{dx} = y(x) V \quad (2)$$

with $z(x) = l_1(x)$ and $y(x) = l_2(x)$, log-periodic functions.

In what follows we shall take up in turn the problem of representation of log-periodic functions, the general solution for a restricted class of log-periodically loaded lines for which a kind of Floquet's Theorem is applicable, and finally a method for finding the input impedance of a general log-periodic transmission line.

3. REPRESENTATION OF $l(x)$.

In discussing the solution of (2), we shall find it useful to have a general series expansion for $l(x)$ which is analogous to the Fourier series for periodic functions. The most elementary log-periodic functions are

$\sin \left(\frac{2\pi \ln x}{\ln \tau} \right)$, $\cos \left(\frac{2\pi \ln x}{\ln \tau} \right)$, and $e^{i \left(\frac{2\pi \ln x}{\ln \tau} \right)}$. If x is replaced by $x\tau$ in the argument of any one of these, the function is unchanged, e.g.,

$$e^{\frac{i 2\pi \ln(x\tau)}{\ln \tau}} = e^{\frac{i 2\pi (\ln x + \ln \tau)}{\ln \tau}} = e^{\frac{i 2\pi \ln x}{\ln \tau}} e^{i 2\pi} = e^{\frac{i 2\pi \ln x}{\ln \tau}}. \quad (3)$$

so that condition (1) is satisfied. This condition is also clearly satisfied if the argument is multiplied by any integer n . In this way an infinite number of different log-periodic functions is generated, one for each integer. This suggests that one may be able to represent an arbitrary log-periodic function by an infinite series of these functions.

3. -- Continued.

$$\ell(x) = \sum_{n=-\infty}^{\infty} a_n x^{\frac{ni 2\pi \ln x}{\ln \tau}} = \sum_{n=-\infty}^{\infty} a_n x^{\frac{in}{a}}. \quad (4)$$

where $a = \frac{1}{2\pi} \ln \tau$. Alternatively,

$$\ell(x) = \sum_{n=0}^{\infty} \left[C_n \sin\left(\frac{n \ln x}{a}\right) + D_n \cos\left(\frac{n \ln x}{a}\right) \right]. \quad (5)$$

We shall see that these series have the same general validity as the Fourier series for periodic functions.

Let us restrict our attention to (4). $U_n(x) = x^{\frac{in}{a}}$ is a solution of the differential equation

$$x^2 u_n'' = x u_n' + \left(\frac{n}{a}\right)^2 u_n = 0. \quad (6)$$

By construction, $u_n(x) = u_n(x\tau)$. The functions $u_n(x)$ form a complete orthogonal set over the period τ , that is, over any interval $b \leq x \leq b\tau$, with b an arbitrary number. The scalar product is complex and requires the weight function $1/x$. The orthogonality may be proved directly.

$$\begin{aligned} \int_b^{b\tau} u_n(x) u_m^*(x) \frac{dx}{x} &= \int_b^{b\tau} x^{\frac{in}{a}} x^{-\frac{im}{a}} \frac{dx}{x} \\ &= \int_b^{b\tau} x^{\frac{i(n-m)}{a} - 1} dx = \ln \tau \delta_{nm}. \end{aligned} \quad (7)$$

δ_{nm} is the Kronecker delta. If we multiply both sides of (4) by $x^{-\frac{im}{a}} dx/x$ and integrate over $b \leq x \leq b\tau$ we find, with the aid of (7), that the coefficients of the expansion in (4) are given by

$$a_n = \frac{1}{\ln \tau} \int_b^{b\tau} \ell(x) x^{-\frac{in}{a}} \frac{dx}{x}. \quad (8)$$

Any function defined over an interval $b \leq x \leq b\tau$ which is square integrable over that interval may be expanded in the log-periodic series according to (5). The $U_n(x)$ are eigenfunctions of the differential

3. -- Continued.

equation (6) with eigenvalues $(\frac{n}{a})^2$. Inasmuch as the interval $(b, b\tau)$ is finite and the series (4) contains all the eigenfunctions of (6), we can reasonably expect that the representation is complete. Of course, any function defined over the finite interval which is expanded according to (5) and (4) becomes a log-periodic function over all x . It is worth emphasizing, however, that for representations of functions over finite intervals, (4) is just as valid as the Fourier series and may even be more useful for some purposes.

As an example, consider the following function in the interval $1 \leq x \leq \tau$.

$$\begin{aligned} f(x) &= A & 0 < x < a \\ &= 0 & a < x < \tau. \end{aligned} \quad (9)$$

According to (8),

$$a_n = \frac{1}{\ln \tau} \int_1^\tau f(x) x^{\frac{-in}{a}} \frac{dx}{x} = \frac{A}{\ln \tau} \int_1^a x^{\frac{-in}{a} - 1} dx = \frac{A(1 - a^{\frac{-in}{a}})}{2\pi in} \quad (10)$$

$$\text{Thus, } f(x) = \frac{A}{2\pi i} \sum_{n=-\infty}^{\infty} \frac{(1 - a^{\frac{-in}{a}})}{n} x^{\frac{in}{a}} \quad (11)$$

It is interesting to consider the limit of the sum (4) when τ is allowed to approach infinity. For this purpose it is convenient to rewrite (4) and (8) slightly.

$$f(x) = \frac{1}{\ln \tau} \sum_{n=-\infty}^{\infty} a_n x^{\frac{in}{a}}; \quad a_n = \int_1^\tau f(x) x^{\frac{-in}{a}} \frac{dx}{x} \quad (12)$$

Proceeding in a formal way, we write

$$\frac{in}{a} = \frac{2\pi in}{\ln \tau} = \nu_n; \quad a_n = a(\nu_n).$$

Then

$$\Delta \nu_n = \frac{2\pi in}{\ln \tau} - \frac{2\pi i(n-1)}{\ln \tau} = \frac{2\pi i}{\ln \tau}.$$

3. -- Continued.

Substituting into (12) we have

$$l(x) = \sum_{n=-\infty}^{\infty} a(v_n) x^{v_n} \frac{\Delta v_n}{2\pi i}. \quad (13)$$

In the limit, as $T \rightarrow \infty$, $\Delta v \rightarrow dv$, and the sum becomes an integral. Thus,

$$l(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} a(v) x^v dv, \text{ with} \quad (14)$$

$$a(v) = \int_1^{\infty} l(x) x^{-v} \frac{dx}{x},$$

and this is just the Mellin Transform. The corresponding operation on the Fourier series turns it into the Fourier integral.

4. THE GENERAL FORM OF SOLUTION FOR A CLASS OF QUASI-LOG-PERIODIC TRANSMISSION LINES.

A transmission line that is periodically loaded may be divided up into cells, all of which are identical. In general, the voltage and current distributions are the same from cell to cell except for a fixed complex multiplier. That is, the voltage and current distributions may be factored into a periodic part, having the period of the structure, and a simple exponential function. This is a statement of Floquet's Theorem, which is generally applicable to differential equations with periodic coefficients.

Let us now consider whether the solution of Equations (2) can in general be factored in a similar way.

$$\frac{dV}{dx} = l_1(x)I \quad ; \quad \frac{dI}{dx} = l_2(x)V. \quad (2)$$

Can the solution to (2) be written as the product of a log-periodic function times a simple function of x ? If a simple factorization is possible, we should expect there to be some cases in which the solution is purely log-periodic. For this to be so, the differential equations, (2), must remain the same when x is replaced by xT .

4. -- Continued.

With this change in (2) we find

$$\frac{dV}{dx} = \tau \ell_1(x) I ; \quad \frac{dI}{dx} = \tau \ell_2(x) V. \quad (15)$$

The equations are altered by the change of variables, and a purely log-periodic solution with $V(x)$, $I(x) = V(x\tau)$, $I(x\tau)$ is not possible.

On the other hand, if the impedance and admittance per unit length are given in the form $\ell(x)/x$, the transmission line equations are invariant under the change of variables, $x = x\tau$.

$$\frac{dV}{dx} = \frac{\ell_1(x)}{x} I ; \quad \frac{dI}{dx} = \frac{\ell_2(x)}{x} V. \quad (16)$$

In terms of the example of Figure 1, a capacitance per unit length of the form $\ell(x)/x$ would represent, insofar as the lumped elements are concerned, capacitors of equal size distributed along a transmission line in a geometric progression. For this type of quasi-log-periodic line it is possible to express the general solution as a simple function of x times a log-periodic function. One can find the form of this solution in much the same manner as in the case of a simply periodic line.

The two equations of (16) can be combined into a single second-order equation for either V or I . Let the two independent solutions of the second-order differential equations be denoted by $g(x)$ and $h(x)$, so that, for example,

$$V(x) = Ag(x) + Bh(x). \quad (17)$$

Clearly, $g(x\tau)$ and $h(x\tau)$ are also solutions, since the change of variables leaves (16) unchanged. Then these must be expressible in terms of $g(x)$ and $h(x)$.

$$g(x\tau) = \alpha_1 g(x) + \alpha_2 h(x), \quad (18)$$

$$h(x\tau) = \beta_1 g(x) + \beta_2 h(x).$$

4. -- Continued.

Then $V(x\tau) = Ag(x\tau) + Bh(x\tau)$

$$= A [a_1 g(x) + a_2 h(x)] + B [\beta_1 g(x) + \beta_2 h(x)] \quad (19)$$

$$= (Aa_1 + B\beta_1)g(x) + [Aa_2 + B\beta_2]h(x)$$

$$= KV(x), \text{ provided that} \quad (20)$$

$$KA = Aa_1 + B\beta_1, \text{ and} \quad (21)$$

$$KB = Aa_2 + B\beta_2.$$

Such a solution other than the trivial one, $A = B = 0$, is possible if K is such that

$$\begin{vmatrix} a_1 - K & \beta_1 \\ a_2 & \beta_2 - K \end{vmatrix} = 0. \quad (22)$$

It is possible that $K=1$, but not in general. We assume that K can always be found to satisfy (22). To see what sort of solution gives rise to the factor K , let

$$f(x) = x^{-\mu} V(x). \quad (23)$$

Then $f(x\tau) = x^{-\mu} \tau^{-\mu} V(x\tau) = x^{-\mu} (\tau^{-\mu} K) V(x) = x^{-\mu} V(x) = f(x)$, provided that $K = \tau^{\mu}$. Thus we can write $f(x) = l(x)$ and

$$V(x) = x^{\mu} l(x). \quad (24)$$

The current distribution must be of the same form.

With the aid of (4) and (24), we can reduce the solution of (16) to the solution of a set of algebraic equations. Let

4. -- Continued.

$$\begin{aligned}
l_1(x) &= \sum_n Z_n x^{\frac{in}{a}} \\
l_2(x) &= \sum_n Y_n x^{\frac{in}{a}} \\
V(x) &= x^\mu \sum_n V_n x^{\frac{in}{a}} \\
I(x) &= x^\mu \sum_n I_n x^{\frac{in}{a}}.
\end{aligned} \tag{25}$$

Substituting these into (16), we find

$$\begin{aligned}
x^\mu \sum_q V_q \left(\mu + \frac{iq}{a}\right) x^{\frac{iq}{a}} &= x^\mu \sum_m Z_m x^{\frac{im}{a}} \sum_q I_q x^{\frac{iq}{a}}, \text{ and} \\
x^\mu \sum_q I_q \left(\mu + \frac{iq}{a}\right) x^{\frac{iq}{a}} &= x^\mu \sum_m Y_m x^{\frac{im}{a}} \sum_q V_q x^{\frac{iq}{a}}.
\end{aligned} \tag{26}$$

Multiplying each equation by $x^{\frac{-in}{a}} \frac{dx}{x}$, and integrating over $(x', x' \tau)$, we obtain the following with the use of (7).

$$\begin{aligned}
V_n \left(\mu + \frac{in}{a}\right) &= \sum_m Z_m I_{n-m}, \text{ and} \\
I_n \left(\mu + \frac{in}{a}\right) &= \sum_m Y_m V_{n-m}.
\end{aligned} \tag{27}$$

If, for example, $Z_n = 0$ for $n \neq 0$, (27) becomes

$$\left(\mu + \frac{in}{a}\right)^2 I_n - Z_0 \sum_m Y_m I_{n-m} = 0. \tag{28}$$

The term μ is found by setting the determinant of the coefficients of the I_n to zero. If this is done with (28), one obtains essentially Hill's determinant, the solution of which is discussed in Whittaker and Watson; μ is given by the expression *

$$\mu = \frac{-4i}{\ln \tau} \sin^{-1} \left\{ [\Delta(0)]^{\frac{1}{2}} \sin \left(\frac{\pi \sqrt{-Z_0 Y_0}}{2} \right) \right\}, \tag{29}$$

* Whittaker and Watson, Modern Analysis, Cambridge, 1927, p. 415.

4. -- Continued.

where the elements of the determinant, $\Delta(0)$, are given by

$$A_{mm} = 1 \quad ; \quad A_{mn} = \frac{Z_0 Y_{m-n}}{4m^2 + Z_0 Y_0} \quad , \quad m \neq n \quad . \quad (30)$$

5. INPUT IMPEDANCE AND ADMITTANCE OF THE LOG-PERIODIC LINE.

Although a general method for solving the transmission-line equation for the quasi-log-periodic line is available (16), there appears to be no corresponding simple procedure for the purely log-periodic case (2). In this section we shall discuss an approximate technique for finding the input impedance or admittance of a semi-infinite non-uniform line and apply the technique to the log-periodic line. The method is an application of the calculus of variations, and depends upon the finding of an expression for the input impedance in terms of an integral over the current distribution such that the expression is stationary with respect to the current distribution.

Consider a non-uniform transmission line extending from $x = 1$ to $x = \infty$, on which the current and voltage are given by

$$\frac{dV}{dx} = z(x) I \quad ; \quad \frac{dI}{dx} = y(x) V \quad . \quad (31)$$

We wish to know what current will flow into the input terminals at $x = 1$ when a known voltage is applied. The ratio of these two quantities is the input impedance,

$$z_i = \frac{V(1)}{I(1)} \quad . \quad (32)$$

In the search for a stationary expression, the best rule is to find an expression which is homogeneous with respect to the distribution. Now,

$$\frac{d}{dx} [VI] = \frac{dV}{dx} I + V \frac{dI}{dx} = Z I^2 + \frac{1}{y} \left(\frac{dI}{dx} \right)^2 \quad . \quad (33)$$

Integrate both sides over $(1, \infty)$.

5. -- Continued.

$$VI \Big|_1^{\infty} = \int_1^{\infty} \left[Z I^2 + \frac{1}{y} \left(\frac{dI}{dx} \right)^2 \right] dx. \quad (34)$$

Let us assume that $|VI|_{x=\infty} = 0$, either due to losses in the line or reflections from internal impedance variations. Actually, this condition must be met if the integral on the right of (34) is to exist. Then,

$$V(1)I(1) = - \int_1^{\infty} \left[Z I^2 + \frac{1}{y} \left(\frac{dI}{dx} \right)^2 \right] dx, \quad \text{and} \quad (35)$$

$$Z_i = \frac{V(1)}{I(1)} = - \frac{1}{[I(1)]^2} \int_1^{\infty} \left[Z I^2 + \frac{1}{y} \left(\frac{dI}{dx} \right)^2 \right] dx. \quad (36)$$

Equation (36) is homogeneous with respect to the current distribution, in the sense that it is independent of the level of $I(x)$. If the correct $I(x)$ is inserted into (36), the exact input impedance will result. Consider now how the computed z_i changes when we insert a slightly erroneous current distribution into (36). That is, let I become $I + \delta I$ and find the resulting δz_i . At $x = 1$, $\delta I = 0$, since we know the input current.

$$\begin{aligned} - [I(1)]^2 \delta Z_i &= \delta \int_1^{\infty} \left[Z I^2 + \frac{1}{y} \left(\frac{dI}{dx} \right)^2 \right] dx = \int_1^{\infty} \left[2 Z I (\delta I) + \frac{2}{y} \left(\frac{dI}{dx} \right) \frac{d(\delta I)}{dx} \right] dx \\ &= \int_1^{\infty} 2 Z I (\delta I) dx + \frac{2}{y} \left(\frac{dI}{dx} \right) (\delta I) \Big|_1^{\infty} - 2 \int_1^{\infty} \frac{d}{dx} \left[\frac{1}{y} \left(\frac{dI}{dx} \right) \right] (\delta I) dx \\ &= 2 \int_1^{\infty} \left\{ Z I - \frac{d}{dx} \left[\frac{1}{y} \left(\frac{dI}{dx} \right) \right] \right\} (\delta I) dx = 2 \int_1^{\infty} \left[Z I - \frac{dV}{dx} \right] (\delta I) dx \\ &= 0. \end{aligned} \quad (37)$$

5. -- Continued.

Thus, $\delta z_i = 0$, showing that to first order the impedance calculated from (36) does not depend on the error in the $I(x)$ used.

Equation (36) may be used to find the exact z_i by successive approximations, and the stationary property guarantees rapid convergence. By the same token, one may obtain a good approximation for z_i by using a simple approximate function for $I(x)$ which is characterized by a few parameters, the parameters chosen so that the resulting impedance be an extremum.

A stationary expression for input admittance may also be found. In (33) we eliminate the current instead of the voltage.

$$\frac{d}{dx} [VI] = \frac{1}{Z} \left(\frac{dV}{dx} \right)^2 + yV^2. \quad (38)$$

$$\text{Then } VI \Big|_1^\infty = + \int_1^\infty \left[\frac{1}{Z} \left(\frac{dV}{dx} \right)^2 + yV^2 \right] dx, \text{ and} \quad (39)$$

$$y_i = \frac{I(1)}{V(1)} = - \frac{1}{[V(1)]^2} \int_1^\infty \left[\frac{1}{Z} \left(\frac{dV}{dx} \right)^2 + yV^2 \right] dx. \quad (40)$$

It is easily shown that $\delta y_i = 0$ with respect to the voltage distribution.

Equations (36) and (40) may be applied to any nonuniform line. Consider the application to the log-periodic line, (2). We first find an approximate expression for the input impedance.

$$\text{Let } \ell_1(x) = \sum Z_n x^{\frac{\ln}{a}}; \quad \frac{1}{\ell_2(x)} = \sum q_n x^{\frac{\ln}{a}}. \quad (41)$$

$$\text{Then } Z_i = - \frac{1}{I^2(1)} \int_1^\infty \left[I^2 \sum Z_n x^{\frac{\ln}{a}} + \left(\frac{dI}{dx} \right)^2 \sum q_n x^{\frac{\ln}{a}} \right] dx. \quad (42)$$

From the results of Section IV, we would expect the most plausible trial function to be: $I(x) = x^\mu$. With this approximation,

$$\begin{aligned} Z_i &\approx - \int_1^\infty \left(x^{2\mu} \sum_n Z_n x^{\frac{\ln}{a}} + \mu^2 x^{2\mu-2} \sum_n q_n x^{\frac{\ln}{a}} \right) dx \\ &= \sum_n \left(\frac{Z_n}{2\mu + \frac{\ln}{a} + 1} + \frac{\mu^2 q_n}{2\mu + \frac{\ln}{a} - 1} \right). \end{aligned} \quad (43)$$

5. -- Continued.

Note that $\text{Re}\{\mu\} < -1/2$ for the integral to exist. The quantity μ must be chosen to make z_i an extremum. To do this we differentiate (43) with respect to μ and set the resulting expression equal to zero, obtaining

$$\sum_n \left\{ \frac{-2 Z_n}{(2\mu + \frac{in}{a} + 1)^2} + \frac{[2\mu(2\mu + \frac{in}{a} - 1) - 2\mu^2]}{(2\mu + \frac{in}{a} - 1)^2} q_n \right\} = 0. \quad (44)$$

One of the roots of (44) is then inserted into (43) for a numerical approximation to z_i .

The corresponding approximation for input admittance may also be found. Let

$$l_2(x) = \sum y_n x^{\frac{in}{a}}, \quad \frac{1}{l_1(x)} = \sum p_n x^{\frac{in}{a}}. \quad (45)$$

With the trial function $V(x) = x^\mu$, (40) becomes

$$y_i \cong \sum_n \left[\frac{y_n}{2\mu + \frac{in}{a} + 1} + \frac{\mu^2 p_n}{2\mu + \frac{in}{a} - 1} \right]. \quad (46)$$

A value of μ which makes (46) an extremum is found from the equation

$$\sum_n \left\{ \frac{-2 y_n}{(2\mu + \frac{in}{a} + 1)^2} + \frac{[2\mu(2\mu + \frac{in}{a} - 1) - 2\mu^2] p_n}{(2\mu + \frac{in}{a} - 1)^2} \right\} = 0. \quad (47)$$

A particular line is characterized by its impedance and admittance per unit length, $l_1(x)$ and $l_2(x)$. Once these are given, the z_n , y_n , q_n , and p_n may be found by means of (8), and the formulas above used to find an approximation to the input impedance and admittance.

6. CONCLUSION.

The next step in the investigation is, clearly, application of the results of Sections IV and V above. Because the solution for the quasi-log-periodic line leads to the Hill determinant for the gross

6. -- Continued.

line transmission factor, $e^{\mu \ln x}$, the properties of such lines may be discussed in terms of standard Brillouin diagrams. Although this is not the case that bears most closely on the antenna problem, it may shed some light on the latter.

On the other hand, the approximate formulae of Section V may be capable of predicting the input impedance or admittance of a log-periodic antenna. Whenever the antenna elements can be approximated by impedance elements on a transmission line, as in the case of the dipole array, (43) and (46) may be employed. Such calculations are currently in progress.

